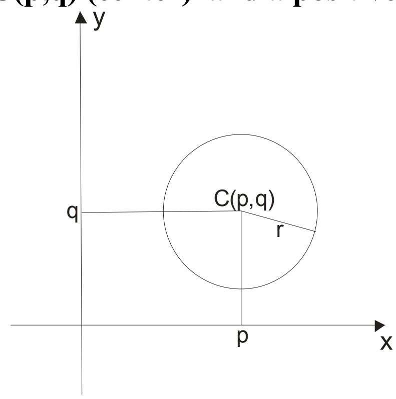
CIRCLE

A **circle** is consisting of those points in a plane which are equidistant from a given point called the center . The common distance of the points of a circle from its center is called its radius.

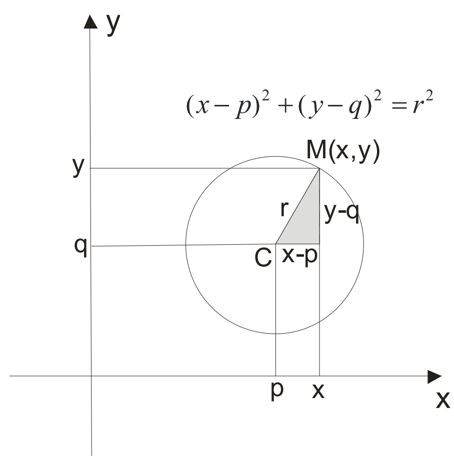
Circle is therefore is given with **point C(p,q) (center) and a positive number r (radius)**



The general equation of the circle is: $(x-p)^2+(y-q)^2=r^2$

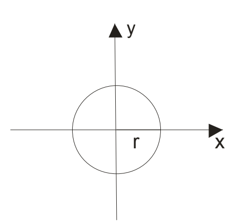
Explanation:

Consider the picture:



Point M(x,y) is on circle . Recognize right-angled triangle in the picture. Application of Pythagorean theorem gives us the required equation of the circle $(x-p)^2+(y-q)^2=r^2$

If p = 0 and q = 0 then it is a **central** circle:



$x^2+y^2=r^2$

There are two ways to find the equation of the circle.

the first way

If the circle is given in the form $x^2 + y^2 + dx + ey + f = 0$ we can use:

$$p = -\frac{d}{2}$$
$$q = -\frac{e}{2}$$
$$r^2 = p^2 + q^2 - f$$

Example 1.

Determine the coordinates of the center and radius of the circle $x^2 + y^2 + 6x - 4y - 12 = 0$

Solution:

Compare $x^2 + y^2 + 6x - 4y - 12 = 0$ with $x^2 + y^2 + dx + ey + f = 0$. We have d = 6, e = - 4 and f = - 12

$$p = -\frac{d}{2} = -\frac{6}{2} = -3$$
$$q = -\frac{e}{2} = -\frac{-4}{2} = 2$$
$$r^2 = p^2 + q^2 - f = (-3)^2 + 2^2 - (-12) = 25$$

Substituting in the equation of the circle $(x-p)^2 + (y-q)^2 = r^2$ we get :

$$(x-p)^2 + (y-q)^2 = r^2$$
$$(x-(-3))^2 + (y-2)^2 = 25$$
$$(x+3)^2 + (y-2)^2 = 25$$

the second way

We complement to the full square!

$$x^2 + y^2 + 6x - 4y - 12 = 0$$

$$x^2 + 6x + y^2 - 4y - 12 = 0$$
$$x^2 + 6x + (\frac{6}{2})^2 - (\frac{6}{2})^2 + y^2 - 4y + (\frac{4}{2})^2 - (\frac{4}{2})^2 - 12 = 0$$
$$\underline{x^2 + 6x + 9} - 9 + \underline{y^2 - 4y + 4} - 4 - 12 = 0$$
$$(x+3)^2 + (y-2)^2 = 25$$

Example 2.

Write the equation of the circle that contains point A(5,6), B(-3,2) and C(-2,-1).

This type of task can be also solved in two ways.

<u>the first way</u>

We use $x^2+y^2+dx+ey+f=0$ and instead of x and y coordinates change given points. Set up a system of three equations with three unknowns and solve it …

$$\begin{array}{lll} A(5,6)\to x^2+y^2+dx+ey+f=0 & B(-3,2)\to x^2+y^2+dx+ey+f=0 & C(-2,-1)\to x^2+y^2+dx+ey+f=0 \\ 5^2+6^2+d\cdot 5+e\cdot 6+f=0 & (-3)^2+2^2+d\cdot(-3)+e\cdot 2+f=0 & (-2)^2+(-1)^2+d\cdot(-2)+e\cdot(-1)+f=0 \\ 5d+6e+f=-25-36 & -3d+2e+f=-9-4 & -2d-e+f=-4-1 \\ 5d+6e+f=-61 & -3d+2e+f=-13 & -2d-e+f=-5 \end{array}$$

Here are three equations, make the system:

$$\begin{array}{l} 5d+6e+f=-61 \\ -3d+2e+f=-13 \\ -2d-e+f=-5 \end{array}$$

We get a solution $d=-4, e=-4, f=-17$ and replace it in $x^2+y^2+dx+ey+f=0$

$$x^2+y^2-4x-4y-17=0$$

$$x^2-4x+(\frac{4}{2})^2-(\frac{4}{2})^2+y^2-4y+(\frac{4}{2})^2-(\frac{4}{2})^2-17=0$$

$$\underline{x^2-4x+4}-4+\underline{y^2-4y+4}-4-17=0$$

$$(x-2)^2+(y-2)^2=25$$

<u>the second way</u>

Given points A(5,6), B(-3,2) i C(-2,-1) directly change in the equation of the circle: $(x-p)^2+(y-q)^2=r^2$

$$\begin{array}{lll} \mathrm{A}(5,6)\to(x-p)^2+(y-q)^2=r^2 & B(-3,2)\to(x-p)^2+(y-q)^2=r^2 & C(-2,-1)\to(x-p)^2+(y-q)^2=r^2 \\ (5-p)^2+(6-q)^2=r^2 & (-3-p)^2+(2-q)^2=r^2 & (-2-p)^2+(-1-q)^2=r^2 \end{array}$$

$$(5-p)^2+(6-q)^2=r^2$$
$$(-3-p)^2+(2-q)^2=r^2$$
$$(-2-p)^2+(-1-q)^2=r^2$$

As in all three equations we have the same right side, compare the left, for example, the first and second, and first and third equation:

$$(5-p)^2+(6-q)^2=(-3-p)^2+(2-q)^2$$
$$25-10p+p^2+36-12q+q^2=9+6p+p^2+4-4q+q^2$$
$$25-10p+36-12q=9+6p+4-4q$$
$$-16p-8q=-48$$
$$2p+q=6$$

$$(5-p)^2+(6-q)^2=(-2-p)^2+(-1-q)^2$$
$$25-10p+p^2+36-12q+q^2=4+4p+p^2+1+2q+q^2$$
$$25-10p+36-12q=4+4p+1+2q$$
$$-14p-14q=-56$$
$$p+q=4$$

We have system of two equations with two unknowns:

$$2p+q=6$$
$$\underline{p+q=4}$$
$$p=2$$
$$q=2$$

Return to one of the first three equations to find the radius r:

$$(5-p)^2+(6-q)^2=r^2$$
$$(5-2)^2+(6-2)^2=r^2$$
$$9+16=r^2$$
$$r^2=25$$
$$r=5$$

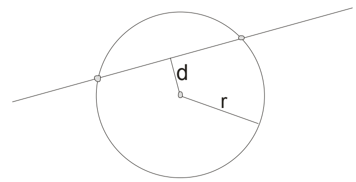
And we get the required equation of the circle:

$$(x-2)^2+(y-2)^2=25$$

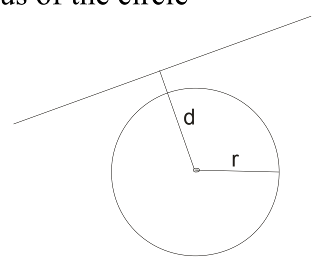
line and circle

For the mutual position of line and circle in the plane , there are three possibilities:

i)
The line and the circles have two common points. This is a situation where the distance from the center of the circle to the line is less than the radius of the circle

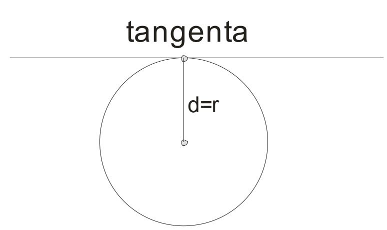


ii)
The line and the circles have not common points. This is a situation where the distance from the center of the circle to the line is greater than than the radius of the circle



iii)

The line and the circles have one common point. This is a situation where the distance from the center of the circle to the line is equal with the radius of the circle. This line is called Tangent.



Investigation of the relationship circle and line is reduced to solving the system of one linear and one quadratic equation

Line: $y = kx + n$ and Circle : $(x-p)^2 + (y-q)^2 = r^2$

i) If $r^2(k^2+1) - (kp - q + n)^2 > 0$ line and circles have two common points
ii) If $r^2(k^2+1) - (kp - q + n)^2 < 0$ line and circles have not common points
iii) If $r^2(k^2+1) - (kp - q + n)^2 = 0$ line and circles have one common point

The situation when line and circle have one common point is also called

CONTACT CONDITION: $r^2(k^2+1)=(kp-q+n)^2$

Note:

If you are looking for a tangent line from point OUTSIDE of the circle, it is necessary to use contact condition.

But if we have to find tangent line from point AT the circle , we use :

$$(x-p)(x_0-p)+(y-q)(y_0-q)=r^2$$

Example 3.

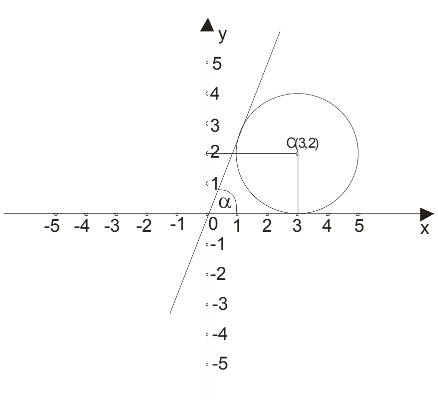
From the point (0,0) are placed tangent to the circle $x^2+y^2-6x-4y+9=0$. *Find their equations and angle between them.*

First, arrange the circles that we can read p,q and r.

$$x^2+y^2-6x-4y+9=0$$
$$x^2-6x+y^2-4y+9=0$$
$$\underline{x^2-6x+9}-9+\underline{y^2-4y+4}-4+9=0$$
$$(x-3)^2+(y-2)^2=4$$

p = 3, q = 2 , r = 2

Draw this circle:



With pictures we can concluded that a tangent line is x- axis , (line y = 0)

Let $y = kx + n$ be our line . Here, change the coordinates of the point from which set tangent line: O(0,0).

$$\begin{aligned} y &= kx + n \\ 0 &= k \cdot 0 + n \\ n &= 0 \end{aligned}$$

We got n = 0.

$k = ?$

$$\begin{aligned} r^2(k^2+1) &= (kp - q + n)^2 \\ 4(k^2+1) &= (3k - 2 + 0)^2 \\ 4k^2 + 4 &= 9k^2 - 12k + 4 \\ 5k^2 - 12k &= 0 \\ k(5k - 12) &= 0 \\ k = 0 &\vee k = \frac{12}{5} \end{aligned}$$

So the required tangent are:

$$t_1 : y = 0$$
$$t_2 : y = \frac{12}{5}x$$

Angle looking over the formula for the angle between lines:

$$tg\alpha = \left| \frac{k_2 - k_1}{1 + k_1 \cdot k_2} \right|$$
$$tg\alpha = \left| \frac{0 - \frac{12}{5}}{1 + 0 \cdot \frac{12}{5}} \right|$$
$$tg\alpha = \frac{12}{5}$$
$$\alpha = arctg\frac{12}{5}$$